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Third Order Compound Option Valuation of Flexible Commodity Based Mining Enterprises

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ABSTRACT

Flexibility in managerial decision making will alter the true value of real world projects. Standard actuarial practice for evaluating real-world projects such as commodity based mining operations rely upon Net Present Value methodology and in essence ignore any flexibility available to the operator to vary the project. Real Option analysis rectifies this to allow better evaluation of economic investment decisions by incorporating managerial flexibility into an option pricing framework. In this paper we extend the results of Konstandatos and Kyng (2012) to evaluate a multi-stage compound mining investment decision where the mining operators have the flexibility to delay project commencement as well as options to abandon production and to expand production to a new mining seam if conditions improve. We allow an independent abandonment of the expansion from the underlying project. We demonstrate that the flexibilities considered give rise to a third-order exotic compound structure, which are evaluated in terms of first, second and third order generalised compound option instruments (Konstandatos (2008)). Our novel representations of the project values contain generalizations of standard compound options and are interpretable as generalised call, call-on-call and call-on-call-on-call type options on the mined commodity price. We provide readily-implementable closed-form analytical formulae which are expressed in terms of the uni-variate, bi-variate and tri-variate Normal distribution functions.

Keywords: Real Options, Commodity Mining Operations, NPV, Risk Neutral Valuation, Exotic Compound Options

INTRODUCTION AND BACKGROUND

Capital budgeting, namely valuation of investment projects, and corporate value creation, are central considerations for investment managers. Resource limitations necessitate accurate valuation and analysis of real-world projects, making managerial flexibility paramount in making decisions in situations with incomplete information. The framework of Real Options naturally arises whenever economic decisions need to be made.

The first author to describe corporate economic assets in terms of financial option considerations was Myers (1977) when examining the determinants of corporate borrowing. Myers identified that the value of a firm reflects an expectation on the firm's future investments. Part of a firm's value consists of the present value of all the options the firm has available to make future investments on favourable terms, contingent upon the decision rule employed to determine whether their managerial 'options' are to be exercised. It was Myers who coined the term 'Real Option' to describe such embedded project flexibilities. Real options may therefore be identified in many industries. Pharmaceutical firms making staged commitments to develop a new drug from concept, through research, manufacture and then marketing is one example. The context considered in the present work, namely mineral exploration and mining operations, is another. In mining operations the real options inherent in the project become apparent where the mining operator is free to decide what circumstances make it worthwhile to commence operations, to expand operations, to delay operations or possibly to abandon existing operations. Real options arise naturally therefore in the development of new mines, in joint ventures and in mineral exploration.

The application of Real Option analysis to commodity based investment operations is a logical extension of traditional capital budgeting methods. In traditional capital budgeting problems the 'discounted cash-flow' model provides the basic framework for most financial analysis. Conventionally the Net Present Value of a project is assumed to be the appropriate measure of the value the project will add to the firm choosing to invest in it. Surveys such as Bhappu and Guzman (1995) and Slade (2001) conclude that discounted cash-flow methods form the basis for investment decisions for most mining companies (Topal (2008)). However, mining operations are extremely capital intensive and usually require many years of production before achieving a positive cash-flow, with a longer project life than many other industries. As observed in Myers (1977), limiting the analysis to discounted cash-flow calculations will tend to understate the project value by ignoring the option value associated with the flexibility to grow profitable lines of business. Dixit and Pindyck (1994) caution that 'the simple net present value rule is not just wrong, it is often very wrong' (p136). The limitations of the discounted cash-flow approach, which fails to consider managerial flexibilities arising from embedded options to delay, expand or abandon a project has led to criticism and to calls for methods which include scope for considering the embedded options when analyzing financial decisions. Fundamentally, in the discounted cash-flow approach there is a failure to allow for the stochastic nature of the output prices. It is this limitation which real option considerations attempt to rectify.

Empirical analysis of investment real options in the mining industry is difficult since the required information is usually private. The empirical study of Moel and Tufano (2002) analysed a private database tracking the opening and closing of 285 developed gold mines in North America in the period 1988-1997. Their analysis of the determinants for commencement and abandonment of mining operations found that the decisions were largely exercised by the mining corporations based on the spot price and volatility of the mined commodity. The study of Colwell et al (2003) analysed the value of the abandonment option for *27* Australian mining companies from 1992-1995 and found that on average the closure option accounted for around *2%* of the individual mine's total value; although these authors cautioned that their conclusions were highly sensitive to assumptions and to input parameters. Bradley (1985) however found limited evidence that mining companies alter their production in light of the movements of the commodity spot price. This study suggested that mining companies make all-or-nothing decisions to commence mining operations and then simply produce at full capacity as long as the spot price exceeds the marginal cost of production. The question of whether mining companies exploit their flexibilities to the fullest extent possible remains open at the moment. It would seem however that many companies underestimate the importance of their available flexibilities to the overall value of their mining operations.
All risk-neutral theories of option pricing, no matter the underlying asset price dynamics, all assume freely traded securities in liquid markets for the underlying asset. The assets underlying the options encountered in many kinds of real options analysis are often not traded

in financial markets. The lack of a readily tradable underlying asset therefore giving rise to objections to the application of modern option pricing theory. Despite this many leading authors argue that it is valid to apply risk neutral valuation approaches to real options situations.

Merton (1998) demonstrated in his Nobel Prize lecture that replication based valuation is still appropriate for pricing derivatives even where replication of the underlying security is not feasible because it is rarely traded. Further, Arnold and Shockley (2002) demonstrated that valuation by no arbitrage pricing principles is the fundamental assumption of both the traditional NPV and the Real Options approaches. In the case of commodity based enterprises the real options based approach can be theoretically justified whenever the value of the project may be expressed as an option on the underlying, liquid and actively traded, commodity.

Brennan and Schwartz (1985). Were amongst the first to apply option pricing theory to mine and oil investment projects. They demonstrated that mining projects could be interpreted and valued as complicated options on the underlying commodities, and used numerical approximation finite difference techniques to perform their evaluations. The analysis of Trigeorgis (1993) also utilised numerical approximation to determine the values of several real option examples via the Binomial pricing method, a well-known numerical approximation scheme for the Black-Scholes framework. This was followed by the first widely available work for practitioners and academics (Trigeorgis (1996)) in which a variety of real option case studies were considered with numerical approximation techniques for their evaluation. Other influential works include Amram and Kulatilaka (1999) and Copeland and Antikarov (2001). More recently Topal (2008) used a decision tree approach with Monte-Carlo simulation in his 'real option' analysis.

In the present paper we take a real options approach which models the stochastic nature of the valuation of commodity-based mining operations using exotic compound option pricing considerations. In our analysis we express the project flexibilities as highly exotic compound options which are priced analytically in our valuation framework. This approach leads to highly symmetric closed form analytic formulae in the Black-Scholes model. We consider mining projects for commodities such as gold and silver, which are also financial assets which are readily traded in highly liquid markets. In effect we demonstrate the valuation of the mining projects under consideration as exotic options on the underlying highly liquid commodity. Before we do that however, it makes sense to give some background to option pricing.

In the standard or plain-vanilla call and put options, the underlying asset is the commodity or stock price itself. Trading in standard options allows the holder to trade and hedge positions in the underlying stock or commodity directly. Compound options in contrast are more complicated instruments where the underlying asset the option is written on is itself another option.

In the standard scenario a compound option confers the right on the holder to trade in a long or short position in another underlying, option. That is, the underlying asset of the compound option is another option contract which references the underlying asset or commodity. For this reason such instruments are sometimes referred to as higher order exotics (Buchen (2004), Konstandatos (2008)). A standard call-on-call compound option for instance will allow the holder to receive a long position in a call option at expiry upon exercise, whereas a call-on-put allows the holder a long position in a put option.

Valuation formulae for European compound options were first developed by Geske (1979). Extensions include the valuation of sequential exchange opportunities (Carr (1988)). Buchen (2004) demonstrates the replication of numerous dual expiry exotic options in terms of standardised 'second-order' instruments, which include as special cases the prices of the call-on-call, call-on-put, put-on-call and put-on-put compound options of Geske. Theoretical methods for the development and pricing of more generalised exotic compound options with both barrier option and lookback option features may be found in Konstandatos (2003, 2008). Lee et al (2008) also apply option pricing theory to evaluate generalised sequential compound options.

The type of compound options we consider here arise naturally in the commodity mining context and may be usefully thought of as non-standard or 'generalised compound options'. We build on Konstandatos and Kyng (2012), which applied similar methods to pricing commodity based mining operations in which results were expressed in terms of dual-expiry (second-order) instruments. The main result of this paper is the pricing of a commodity-based mining project with the flexibility to delay, expand and abandon operations, in which the expanded operations themselves have the added flexibility of abandonment after commencement, requiring the use of first, second and third order generalised compound option instruments. To do this we demonstrate the decomposition of our project valuations into first, second and third-order generalizations of the Gap-option instruments as defined in Section 2.

The remainder of this paper is structured as follows. Section 2 provides an overview of option pricing theory, and sets up the notation and framework which we employ in our analysis This methodology is an extension of Buchen (2004) to the tri-expiry scenario, and forms the non-standard methodology which we utilize to price the exotic option structures appearing in this paper. Section 3 contains the main contribution of this paper. Section 3.1 provides a succinct closed-form analytic expression for a 'basic project' with delay and abandonment. Our valuation is expressed solely in terms of one first order Gap instrument and one second order Gap option instrument, which are interpretable respectively as a generalised call option and a generalised call-on-call type exotic compound option on the underlying commodity price. This is a new representation for the value of such a project, and agrees with the previously published formulae in Konstandatos and Kyng (2012). Section 3.2 contains the valuation of the 'compound mining project' in which we add an option to expand production, with the expansion itself having its own option to abandon. Our closed form valuation formula requires the tri-expiry valuation framework and the use of our third-order generalisations of the Gap option instruments from Section 2. The appearance of the additional terms up to third-order in the project value are then interpretable as the contribution of 'call-on-call' and 'call-on-call-on-call' compound options on the commodity to the project price. Section 4 provides numerical valuation and discussion of our formulae followed by a brief conclusion in Section 5.

OPTION PRICING FRAMEWORK

For completeness this section gives an overview of option pricing theory in the Black-Scholes framework we're working in, and a description of the notation which will be utilised our analysis in Section 3. A readable introduction to the mathematics of option pricing theory may be found in Wilmott et al (1995).

Review of Option Pricing Theory

The Black-Scholes Option Pricing model, building on Samuelson (1965), was developed in the early 1970s and is now considered a classic result in the Finance industry. Using the idea of efficient markets, Black and Scholes (1973) and Merton (1973) demonstrated that an option over a stock has an economic value depending on x, (the market price of the stock) and t, (the time elapsed since the option was written). The model assumes the stock price process is geometric Brownian motion, and makes several idealized assumptions about market frictions. The underlying stock process is assumed to follow the dynamics described by the Log-Normal Stochastic Differential Equation:

$$dX_t = (\mu - q)X_t dt + \sigma X_t dW_t \quad t \geq 0 \tag{1}$$

The parameters (μ, q, σ) represent the drift, continuous dividend yield and the volatility of the asset. W_t Represents a standard Wiener Process. In the mining context q will represent the yield of the underlying asset net storage costs which are assumed to be proportional to the spot price.

Let $V(x,t)$ be the value at time t value of some option contract defined over an asset with current value x. $V(x,t)$ Satisfies the Black-Scholes Partial Differential Equation (PDE) on the domain $D = \{(x,t)|x > 0, 0 < t < T\}$, subject to the terminal boundary condition $V(x,T) = f(x)$, where $f(x)$ is any payoff-function of the stock price x.

$$\frac{\partial V}{\partial t} + (r - q)x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - rV = 0 \tag{2}$$

Where r is the risk-free rate. The time t=T is the 'option maturity'. A European call of strike price K has payoff $f(x) = \max(x - K, 0) = (x - K)^+$ and satisfies $V(0,t) = 0$, namely the option value is zero if the asset becomes worthless. The European put option satisfies similar conditions. The solutions subject to the relevant boundary and expiry conditions for European call and put options were shown by Black and Scholes (1973) to be given by:

$$\begin{aligned} BSCall(x,K,r,q,\sigma,\tau) &= xe^{-q\tau}N(d_1) - Ke^{-r\tau}N(d_2) \\ BSPut(x,K,r,q,\sigma,\tau) &= Ke^{-r\tau}N(-d_2) - xe^{-q\tau}N(-d_1) \end{aligned} \tag{3}$$

Where $[d_1, d_2] = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + (r - q \pm \frac{1}{2}\sigma^2)\tau\right]$ and $\tau = T - t$.

The PDE approach for option pricing was first used in Black and Scholes (1973) to derive analytic option pricing formulae. PDE methods also are the basis for various discretization schemes such as finite differences for numerical approximation methods. An alternative approach or option pricing is due to Harrison and Pliska (1981). This approach determines the option price by computing the expected option payoff under the equivalent martingale measure (also known as the *risk neutral distribution*), discounted at the risk free rate of interest. This amounts to setting the dirft the risk-free rate, $\mu = r$ in Eq (1). The risk neutral expectations method is mathematically equivalent to solving PDE (1) subject to relevant boundary conditions. This follows from the Feynman-Kac formula which relates the solutions of parabolic Linear PDE boundary value problems to quadratures against densities satisfying the forward and backward Kolmogorov equations defining certain transition probability density functions (Kac (1949); Konstandatos (2008) has further details).

Numerical methods are typically applied when it is difficult or impossible to derive analytical valuation formulae. Various numerical methods exist for option pricing, most notably Monte Carlo simulation (Boyle (1977)) and the binomial method (Cox, Ross and Rubinstein (1979)). The binomial method is a discrete time, discrete state space approximation which models the asset price distribution as "log-binomial" rather than log-normal. Another approach is to apply finite difference methods for parabolic PDEs to the Black-Scholes equation (Hull and White (1990)). Examples of the use of numerical methods in the real options context can be found in Trigeorgis (1993, 1996).

Generalised compound options

In this section we define and provide formulae for the non-standard instruments which we use to represent our prices. The third-order instruments which we define below are the generalisations to a tri-expiry framework of a dual-expiry pricing methodology outlined in Buchen (2004). The 'compound' nature of our instruments arises from the payoffs which define them. The holder of a generalised compound instrument is to receive another (underlying) generalised compound instrument of lower-order with a longer expiry date rather than the stock or commodity, provided the exercise condition is satisfied.

The most basic instrument we consider the first-order 'Gap-option'. This isn't a compound option, but it is defined by its payoff at some time$t = T$. For reasons which will become apparent we think of this as a 'generalised first-order instrument'. It coincides with standard call and put options for specific choices of parameters.

$$G^{s}_{\xi\,;K}(X)|_{t=T} = (X - K)\mathbf{1}(s\,X > s\,\xi) \qquad (4)$$

We have used the statistical *indicator function* $\mathbf{1}(.)$ (*i.e.* the Heaviside step function) in the payoff to specify the exercise condition on the underlying asset at expiry:

$$\mathbf{1}(X \gtrless 0) = \begin{cases} 1 & \text{if } X \gtrless 0 \\ 0 & \text{otherwise} \end{cases}$$

The upper index $s = \pm 1$ in the definition is used to specify the direction of the greater-than/less-than sign in the exercise condition for the asset price X in the Gap option payoff. It should be clear that the above specification defines a generalised option with a decoupling of the exercise price ξ from the strike price$K$. The particular choice of exercise condition $s = +1$ and exercise price $\xi = K$ coincides with a plain-vanilla call option of strike price $K,$ whereas $s = -1, \xi = K$ reproduces the formula for a put option up to sign. Note also the upper and lower index pairing of the exercise indicator with exercise price. Closed form formulae are readily obtained in the Black-Scholes framework (see Konstandatos (2003, 2008):

$$G^{s}_{\xi\,;\,K}(x,\tau) = x\,e^{-q\,\tau}N\big(d_{\xi}\big) - \,K\,e^{-r\tau}N\big(d_{\xi} - \sigma\sqrt{\tau}\big) \qquad (5)$$

Where $d_{\xi} = \frac{1}{\sigma\sqrt{\tau}}\Big[\ln(x/\xi) + (r - q + \frac{1}{2}\sigma^2)\tau\Big]$. In the above notation we may express the prices of standard call and put European options as follows, in agreement with Eqs (3):

$$\begin{aligned} BSCall(x,K,r,q,\sigma,\tau) &= \;\; G^{+}_{K\,;\,K}(x,\tau) \\ BSPut(x,K,r,q,\sigma,\tau) &= -G^{-}_{K\,;\,K}(x,\tau) \end{aligned}$$

Where $\tau = T - t$ is the time remaining to expiry? The above instrument has one expiry time, $t = T$ which corresponds to the option payoff.

Second-order instruments are defined with two future expiry times, (T_1, T_2) with $T_1 < T_2$. At expiry time T_1 this instrument pays a first-order Gap option as defined in Eq (5) with strike price K , index condition / exercise price ($s_{2,}\xi_2$) and expiry time at time T_2 provided the underlying asset exercise condition $s_1\,X > s_1\,\xi_1$ is satisfied at time T_1. The exercise condition at time T_1 is equivalent to the underlying asset being either above or below the exercise price ξ_1 when $s_1 = \pm 1$ respectively.

$$G_{\xi_1\,\xi_2\,;K}^{S_1\,S_2}(X)|_{t=T_1} = G_{\xi_2\,;K}^{S_2}\,(X, T_2 - T_1)\mathbf{1}(s_1\,X > s_1\,\xi_1) \qquad (6)$$

The closed-form analytic expression for the price of this instrument at time $t < T_1$ is determined by expectations. The result is:

$$\begin{aligned} G_{\xi_1\,\xi_2\,;K}^{S_1\,S_2}&(x,\tau_1,\tau_2) \\ &= x\,e^{-q\,\tau_2}N_2\left(d_1, d_2; s_1 s_2\sqrt{\frac{\tau_1}{\tau_2}}\right) \\ &- K\,e^{-r\,\tau_2}N_2\left(d_1', d_2',; s_2\sqrt{\frac{\tau_1}{\tau_2}}\right) \end{aligned} \qquad (7)$$

Where

$$[d_1, d_1'] = \frac{1}{\sigma\sqrt{\tau_1}}\left[\ln\left(\frac{x}{\xi_1}\right) + \left(r - q \pm \frac{1}{2}\sigma^2\right)\tau_1\right]$$
$$[d_2, d_2'] = \frac{1}{\sigma\sqrt{\tau_2}}\left[\ln\left(\frac{x}{\xi_2}\right) + \left(r - q \pm \frac{1}{2}\sigma^2\right)\tau_2\right]$$

$N_2(x, y; \rho)$ Denotes the bi-variate Normal distribution function with correlation coefficient ρ. The third-order generalised compound instruments are defined with three future expiry times, (T_1, T_2, T_3) with $T_1 < T_2 < T_3$. At expiry time T_1 this instrument pays a second-order Gap option as defined in Eq (7) with strike price K and index conditions / exercise prices ($s_{1,}\xi_1$), ($s_{2,}\xi_2$) and dual expiry times (T_2, T_3), provided the underlying asset exercise condition $s_1\,X > s_1\,\xi_1$ is satisfied at time T_1.

$$\begin{aligned} G_{\xi_1\,\xi_2\,\xi_3\,;\,K}^{S_1\,S_2\,S_3}&(X)|_{t=T_1} \\ &= G_{\xi_1\,\xi_2\,;K}^{S_1\,S_2}\,(X, T_2 - T_1, T_3 - T_1)\mathbf{1}(s_1\,X > s_1\,\xi_1)\,\mathbf{1}(s_2\,X \\ &> s_2\,\xi_2) \end{aligned} \qquad (8)$$

It is also possible to derive the following closed-form analytic expression for this third-order abstract instrument in terms of the tri-variate Normal distribution. (See Konstandatos (2008) for details of the calculations).

$$G_{\xi_1\,\xi_2\,\xi_3\,;\,K}^{S_1\,S_2\,S_3}(x,\tau_1,\tau_2,\tau_3) = x\,e^{-q\,\tau_3}N_3(\boldsymbol{d},\boldsymbol{\Omega}) - K\,e^{-r\tau_3}N_3(\,\boldsymbol{d}',\boldsymbol{\Omega}) \qquad (9)$$

Where $= \begin{pmatrix} \frac{1}{\sigma\sqrt{\tau_1}}\left[\ln\left(\frac{x}{\xi_1}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)\tau_1\right] \\ \frac{1}{\sigma\sqrt{\tau_2}}\left[\ln\left(\frac{x}{\xi_2}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)\tau_2\right] \\ \frac{1}{\sigma\sqrt{\tau_3}}\left[\ln\left(\frac{x}{\xi_3}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)\tau_3\right] \end{pmatrix}$, $\boldsymbol{d}' = \begin{pmatrix} \frac{1}{\sigma\sqrt{\tau_1}}\left[\ln\left(\frac{x}{\xi_1}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)\tau_1\right] \\ \frac{1}{\sigma\sqrt{\tau_2}}\left[\ln\left(\frac{x}{\xi_2}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)\tau_2\right] \\ \frac{1}{\sigma\sqrt{\tau_3}}\left[\ln\left(\frac{x}{\xi_3}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)\tau_3\right] \end{pmatrix}$

$N_3(\boldsymbol{d};\boldsymbol{\Omega})$ Denotes the tri-variate Normal distribution function with arguments $\boldsymbol{d}$ and with 3-D correlation matrix, which depend on the overlapping times to the three expiry dates:

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & s_1 s_2\sqrt{\frac{\tau_1}{\tau_2}} & s_1 s_3\sqrt{\frac{\tau_1}{\tau_3}} \\ s_1 s_2\sqrt{\frac{\tau_1}{\tau_2}} & 1 & s_2 s_3\sqrt{\frac{\tau_2}{\tau_3}} \\ s_1 s_3\sqrt{\frac{\tau_1}{\tau_3}} & s_2 s_3\sqrt{\frac{\tau_2}{\tau_3}} & 1 \end{pmatrix}$$

In the expressions for the second-order and third-order Gap option formulae (Eqs (7, 9)) we have denoted $\tau_i = T_i - t$ for $i = 1,2,3$ to be the durations to the expiry times T_i respectively. Theξ_i, $i = 1,2,3$ are the generalised exercise prices for the correspondingT_i. It is the existence of the multiple expiry times and exercise prices in the definitions of our instruments that give rise to the 'dual–expiry' and 'tri-expiry' structure in our analysis.

STRUCTURE OF COMMODITY BASED MINING PROJECT

In this section we recap the essential features of the commodity project framework set up in Konstandatos and Kyng (2012) before extending the real option analysis to evaluate a more complicated version of the mining operations considered in that paper. It is useful to think of the mined commodity to be gold, as it is a mineral, a commodity and an investment asset actively traded in financial markets. The market price is readily observable, along with gold futures prices and the prices of standard options and other financial derivatives. Our analysis however is applicable to any commodity based project. It transpires that all the project variants considered in this paper have closed form analytic values which are expressible as options on the commodity price, in terms of the exotic Gap option instruments described in Section 2.

Flexible project with option to delay and option to abandon.

In this section we derive a new closed-form analytical formula for the basic project of this paper, namely a project with an option to delay commencement with another option to abandon the project at some future time after commencement for salvage value. To make the ideas concrete, suppose the project sponsor has the option to commence some gold mining project at some future timeT_0. By deciding to invest, the sponsor must outlay an initial amount of capitalK_0. In return, profits are received at times $T_1, T_2, \ldots, T_n$ of amount $X_{T_i} - C$ at time T_i respectively, where $X_{i_i}$is the time T_i market price of gold and C is the cost of extracting and processing the gold each period. We assume the cost of extraction is constant, and that by committing the sponsor will be locked-in to the project and the cashflows until such time T_m at which the sponsor has the right to abandon the project if conditions have worsened (say with the a substantial drop in the commodity price) for some salvage value S_m. We will refer to the structure outlined above as the 'basic project' which will serve as the building block for the extra flexibility to be considered in the next section.

It is a basic financial result arising from arbitrage considerations that the expected commodity price is the forward price for commodities which are also investment assets such as gold and silver. Such assets provide owners with income as well as incurring storage costs. Let q denote the net storage costs (convenience yield) calculated proportionally to the spot price, with the risk-free rate *r*. For times $T_i > T_0$ we have that

$$E\{X_i|X_0\} = e^{(r-q)(T_i-T_0)}X_0$$

Following Konstandatos and Kyng (2012) we define the annuity factor for given integer parameters *(r,m,n)* as follows:

$$A(r,m,n) = \sum_{i=1}^{n} e^{-r(T_i-T_m)}$$

With this annuity factor and the relationship between spot and forward prices we obtain the following expression for the discounted expected cash-flows from times T_0 to T_n in terms of the commodity price at time T_0 and the fixed cost of extraction per period$C$:

$$\sum_{i=1}^{n} e^{-r(T_i-T_0)}E\{X_{T_i} - C|X_{T_0}\} = A(q,0,n)X_0 - C\,A(r,0,n)$$

We now turn our attention to consider a project with an option to delay commencement until some future time T_0 with an added flexibility to abandon the project at time some time $T_m > T_0$ at which time the mining operators may recoup some salvage value for the project S_m for the abandoned operations before the project's end at time T_n.

Consider the foregone cashflows at time T_m. clearly the mining operator will rationally choose to abandon the project provided that the salvage value exceeds these foregone cashflows. Since we have

$$PV_{T_m}\{\text{Foregone cashflows}\} = A(q,m,n)\,X_m - C\,A(r,m,n)$$

it follows the mining operator will choose between the greater:

$$\begin{aligned} PV_{T_m}\{\text{Project}\} &= \max\big(S_m\,, A(q,m,n)X_m - C\,A(r,m,n)\big) \\ &= S_m + (A(q,m,n)X_m - C\,A(r,m,n) - S_m)^+ \end{aligned}$$

This is readily seen to be a call option on the time T_m commodity price. The project value at time T_0 is therefore expressible as

$$\begin{aligned} E_{T_0}\{NPV\{Project\}\} &= A(q,0,m)X_0 - C\,A(r,0,m) - K_0 + S_m e^{-r(T_m-T_0)} \\ &+A(q,m,n)G^{+}_{K';K'}(X_0,T_m-T_0) \end{aligned}$$

Where K' is defined in Eq (10). The reader should observe that the last term $G^{+}_{K';K'}(X_0,T_m - T_0)$ defines a call option on the commodity price with time $T_m - T_0$ to expiry and with strike price K'. with the further definition of the exercise price K''in Eq (10),

$$K' = \frac{C\,A(r,m,n) + S_m}{A(q,m,n)};\; K'' = \frac{C\,A(r,0,m) + K_0 - S_m e^{-r(T_m-T_0)}}{A(q,0,m)} \qquad (10)$$

We write the value of the project as a strictly increasing function of the commodity price *x* at timeT_0.

$$Val\{Proj\}_{T_0}(x) = A(q,0,m)(\,x - K'') + A(q,m,n)G^{+}_{K';K'}(x,T_m - T_0) \qquad (11)$$

We may solve this equation numerically for the critical price $x = a$ for which $Val\{Proj\}_{T_0}(x) > 0$ whenever $x > a$ and vice-versa. In terms of the critical price the project value for times before commencement, in terms of the currently observed commodity price $X_t = x$ is expressible solely in terms of one first order instrument with exercise/strike $(a; K'')$ and one second order instrument of generalised exercise prices (a, K') and strike priceK':

$$Val\{Proj\}_{t<T_0}(x) = A(q,0,m)G^{+}_{a\,;\,K''}(x,T_0-t) + G^{++}_{aK';K'}(x,T_0-t,T_m-t) \quad (12)$$

Note this very brief representation of the project value is deceptively simple. The complexity of the underlying structure is subsumed into the definitions of the first order and second order instruments which we utilised from the framework in Section 2. When expanded fully the representation of the project value requires first and second order instruments expressible in terms of the uni-variate and bi-variate cumulative Normal Distribution functions respectively. It is relatively straightforward to demonstrate that this representation agrees with Konstandatos and Kyng (2012) when fully expanded. The utility of the above representation will become apparent in the next section, where we consider a more complicated project with option to delay commencement with abandonment, with a further option to expand production, where the project expansion also comes with its own option to abandon, mirroring the flexibility a mining operator may have to expand production and scale the expansion back independently of an ongoing underlying project.

Flexible compound project: flexible project with a flexible expansion.

In this section we will extend the results of the previous section to evaluate a project with the added right to expand production at some future stage after project commencement. We maintain the basic structure from the previous section, namely an underlying mining project with a right to defer commencement to a future time $T_0$with the right to abandon at timeT_m. Added to this is the further operating flexibility to expand mining production at some time T_p with different underlying cost structures.

To commence the overall project the sponsor must outlay an initial amount of capital K_0.at time T_0 In return, the mining operator will receive profits at times $T_1, T_2, \dots, T_n$ of amount $X_{T_i} - C_0$ at time T_i respectively, where $X_{i_i}$is the time T_i market price of the commodity and C_0 is the cost of extracting and processing each period. Furthermore, if the mining operator chooses to expand mining production at time T_p say, they will receive profits at times $T_{p+1}, T_{p+2}, \dots, T_{n'}$ of amounts $X_i - C_p$ where C_p represents the (possibly higher) cost of extracting the ore from the expanded operations, and where time $T_{n'}$ is the end-time for the expanded component of the project. Furthermore, we will allow that the mining operator has the option to abandon the expanded component of the project if they so choose at some time$T_{m'} < T_{n'}$. We also allow the added flexibility that the expanded project component may be abandoned without necessarily abandoning the basic project. We introduce this structure with following rationale. As prices rise for the underlying commodity it may make it profitable for the mining operator to expand production and mine ore which is more difficult and costly to mine, possibly by sinking a deeper shaft or by commencing operations in another costlier location. The flexibility in mining the costlier ore is considered independently from mining the less costly ore in the basic project. It should be apparent to the reader that the basic structure we are considering is of a mining project with an option to delay commencement and with the option to abandon at some time after commencement, as well as a further set of options to expand mining operations if the commodity price improves, with the expansion itself having an option to abandon if conditions subsequently worsen. The pricing of this project demonstrates the higher-order compounding effects in the decision process, requiring the appearance of first, second and third order cumulative Normal distribution functions in the framework we're utilising from Section 2.. The extension of the pricing formulae in Konstandatos and Kyng (2012) to the pricing of this *flexible project* with a *flexible expansion* is the basic new result of this paper.

To begin our analysis we consider the project at timeT_0. The expanded component of the project at this point in time may be considered as a project with an option to delay commencement with abandonment at times $(T_p, T_{m'})$ respectively. From the representation in Eq (12) we can write the value of the expanded project as a function of the time T_0 commodity price X_0:

$$Val\{Proj\ E\}_{T_0}(X_0) = \\ A(q,p,m')G^{+}_{a';\,K''_E}(X_0, T_0 - t) + A(q,m',n')G^{++}_{a'K'_E\,;\,K'_E}\left(X_0, T_p - T_0, T_{m'} - T_0\right)$$

Where $K'_E, K''_E$the generalised compound exercise are prices and where $x = a'$ is the critical value solution of

$$A(q,p,m')(\,x - K''_E) + A(q,m',n')G^{+}_{K'_E\,;\,K'_E}\left(x, T_{m'} - T_p\right) = 0 \qquad (13)$$

Assuming that K_p is the fixed cost for commencement of the expansion component in the event the option to expand production is exercised, the following expressions follow for the generalised exercise prices:

$$K'_E = \frac{C_p A(r,m',n') + S_{m'}}{A(q,m',n')};\ K''_E = \frac{C_p\, A(r,p,m') + K_p - S_{m'}e^{-r\left(T_{m'} - T_p\right)}}{A(q,0,n)} \qquad (14)$$

The expected NPV of the whole project at time T_0 is therefore:

$$E\{NPV\{Proj\}\}_{T_0}(X_0) = \\ A(q,0,m)X_0 + C_0A(r,0,m) - K_0 + S_m e^{-r(T_m - T_0)} \\ + A(q,m,n)G^{+}_{K'_0\,;\,K'_0}(X_0, T_m - T_0) + A(q,p,m')G^{+}_{a';\,K''_E}(X_0, T_0 - t) \\ + A(q,m',n')G^{++}_{a'K'_E\,;\,K'_E}\left(X_0, T_p - T_0, T_{m'} - T_0\right)$$

Where K_0 is the fixed cost of commencing the whole project at timeT_0. This expression consists of the project cashflows from time T_0 to abandonment time T_m and the T_0 value of the expansion component of the project. we can write the T_0 expected NPV in the symmetric form:

$$E\{NPV\{Proj\}\}_{T_0}(X_0) = \\ A(q,0,m)\,(\,X_0 - K''_0) + A(q,m,n)G^{+}_{K'_0\,;\,K'_0}(X_0, T_m - T_0) \\ + A(q,p,m')G^{+}_{a';\,K''_E}(X_0, T_0 - t) \\ + A(q,m',n')G^{++}_{a'K'_E\,;\,K'_E}\left(X_0, T_p - T_0, T_{m'} - T_0\right) \qquad (15)$$

Where we have introduced the exercise prices

$$K'_0 = \frac{C_0 A(r,m,n) + S_m}{A(q,m,n)};\ K''_0 = \frac{C_0\, A(r,0,m) + K_0 - S_m e^{-r(T_m - T_0)}}{A(q,0,m)} \qquad (16)$$

Each term in Eq (15) for the expected NPV is a strictly increasing function of X_0 which may be zero. The rational mining operator will proceed when the commodity price exceeds some threshold value $X_0 = a$, and will not otherwise proceed with the project. In terms of this critical value, we can we see that the whole project can be thought of as a complicated exotic option on the commodity price at time T_0, with $t < T_0$ price given in terms of four discounted expectations:

$$\begin{aligned} e^{r(T_0-t)}Val\{Proj\}_{t<T_0}(x) \\ &= E_Q\{\,A(q,0,m)\,(\,X_0 - K_0'')\mathbf{1}(X_0 >\ a)|X_t = x\} \\ + A(q,m,n)\,E_Q\{\,G^{+}_{K_0'\,;\,K_0'}(X_0,T_m - T_0)\mathbf{1}(X_0 >\ a)|X_t = x\} \\ + A(q,p,m')E_Q\{G^{+}_{a';\,K_E''}\big(X_0,T_p - T_0\big)\mathbf{1}(X_0 >\ a)|X_t = x\} \\ +\ A(q,m',n')\,E_Q\{\,G^{++}_{a'K_E'\,;\,K_E'}\big(X_0,T_p - T_0,T_{m'} - T_0\big)\mathbf{1}(X_0 >\ a)|X_t = x\,\} \end{aligned} \tag{17}$$

The expectations are conditional on $X_t = x$, the observed time t commodity spot-price. Note we have taken the exponential discount factor to the left hand side in Eq (17). Given the numerical critical values (a, a') obtained via numerical solution of Eq (15) and Eq (13) respectively, under the framework of Section 2 .we obtain the following compact expression for the value of the project conditional on the currently observed commodity price x :

$$\begin{aligned} Val\{Proj\}_{t<T_0}(x) \\ &= A(q,0,m)\,G^{+}_{a;K_0''}(x,T_0 - t) \\ &+\ A(q,m,n)\,G^{++}_{aK_0';K_0'}(x,T_0 - t,T_m - t) \\ &+ A(q,p,m')\,G^{++}_{aa';\,K_E''}\big(x,T_0 - t,T_p - t\big) \\ &+ A(q,m',n')\,G^{+++}_{aa'K_E'\,;\,K_E'}\big(x,T_0 - t,T_p - t,T_{m'} - t\big) \end{aligned} \tag{18}$$

In the order appearing above, we see that the value of the project depends on one first-order instrument on the commodity price expressible in terms of the uni-variate Normal distribution, two second-order instruments expressible in terms of the bi-variate Normal distribution; and one third-order instrument expressible in terms of the tri-variate Normal distribution. These instruments may be interpreted respectively as follows: a generalised call option on the commodity price with exercise prices and strike prices (a, K_0''); two generalised call–on–call type options on the commodity price with generalised exercise prices and strike prices (a, K_0') and (a, a', K_E'') respectively; and finally one generalised call-on-call-on-call type option on the commodity price with generalised exercise prices and strike (a, a', K_E'). In the above scenario we have assumed that $T_m \geq T_p$. the case $T_m < T_p$ involves a more complicated decision structure requiring fourth order instruments which we do not cover here.

NUMERICAL IMPLEMENTATION AND DISCUSSION

In this section we present numerical valuations of Eq (12) and Eq (18). The calculations were implemented in Matlab. The algorithm of Drezner (1989) was used to compute the bi-variate normal cumulative density function (also documented in Hull (2009)). The tri-variate normal cumulative density function was implemented using the quasi-monte Carlo algorithms documented in Gentz (1993).

Project 1, is a 'basic flexible project' of Section 3.1 (Eq (12)). It has an option to commence gold mining operations in one year, with 3-monthly commodity extraction for 6 years with an option to abandon after 3.5 years for salvage value S_m . The extraction costs are

C_0 =\$800/ounce with a fixed cost of commencement of \$2mil and salvage value of \$1mil. The parameters used are summarised in the table below for risk-free rate r, convenience yield y and commodity volatility:

Table 1: Parameters for Project 1.

T_0	T_m	T_n	K_0	S_m	C_0	r	y	σ
1yr	3.5yrs	6yrs	\$2,000,00(	\$1,000,00(	\$800	0.10	0.02	0.15

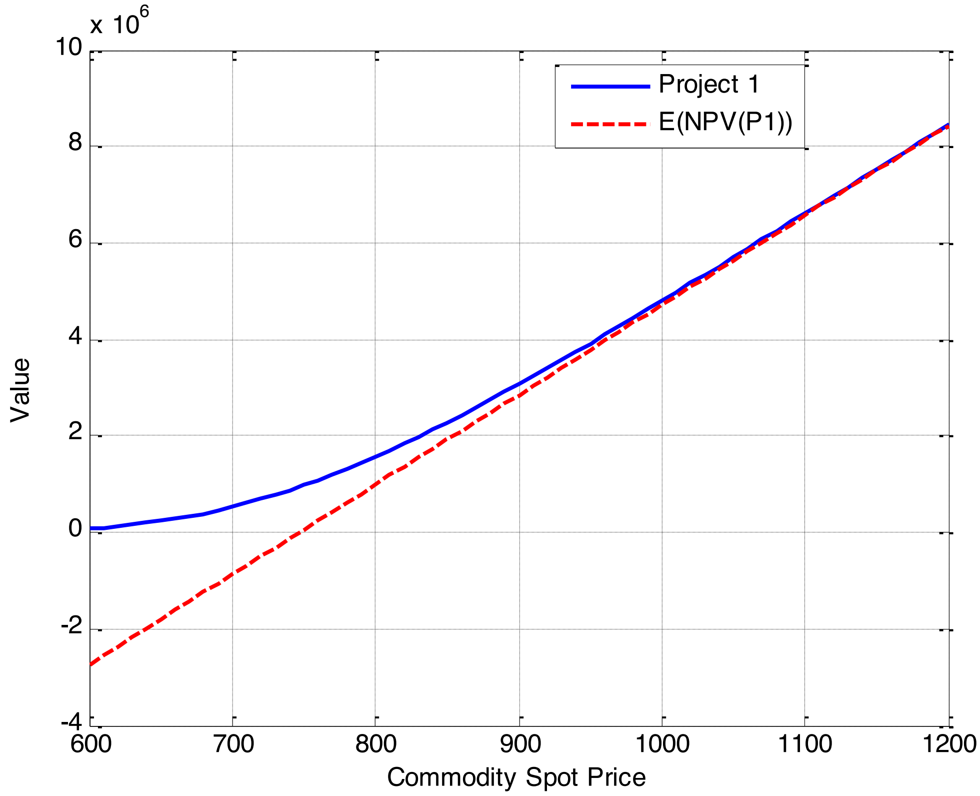

Figure 1. Project 1. Mining project with delay and abandonment / expected NPV.

Project 2 corresponds to the flexible project of Section 3.2 (Eq (18)) and consists of Project 1 (and with an option to delay commencement of all operations untilT_0) with a further option to expand production 1 year after commencement for 3 years. The option to abandon the expanded operations is taken to occur one year after the commencement of the expansion. The expansion also has 3-monthly commodity extraction for the maximum 3 year life with higher extraction costs of C_p per ounce, a lower salvage value for the expanded component $S_{m'}$ and a higher fixed cost of commencementK_p. The parameters for Project 2 are otherwise the same as in Table 1 with the extra requirements summarized in Table 2:

Table 2: Parameters for Project 2.

$T_{m'}$	$T_{n'}$	K_p	$S_{m'}$	C_p
3yrs	4yrs	\$3,000,000	\$750,000	\$1100/ounce

Matlab's numerical solver was used to solve the transcendental equation Eq (11) obtain the critical value $x = a$ for Project 1 and to numerically solve the transcendental Eqs (13, 17) for $x = (a', a)$ respectively for Project 2.

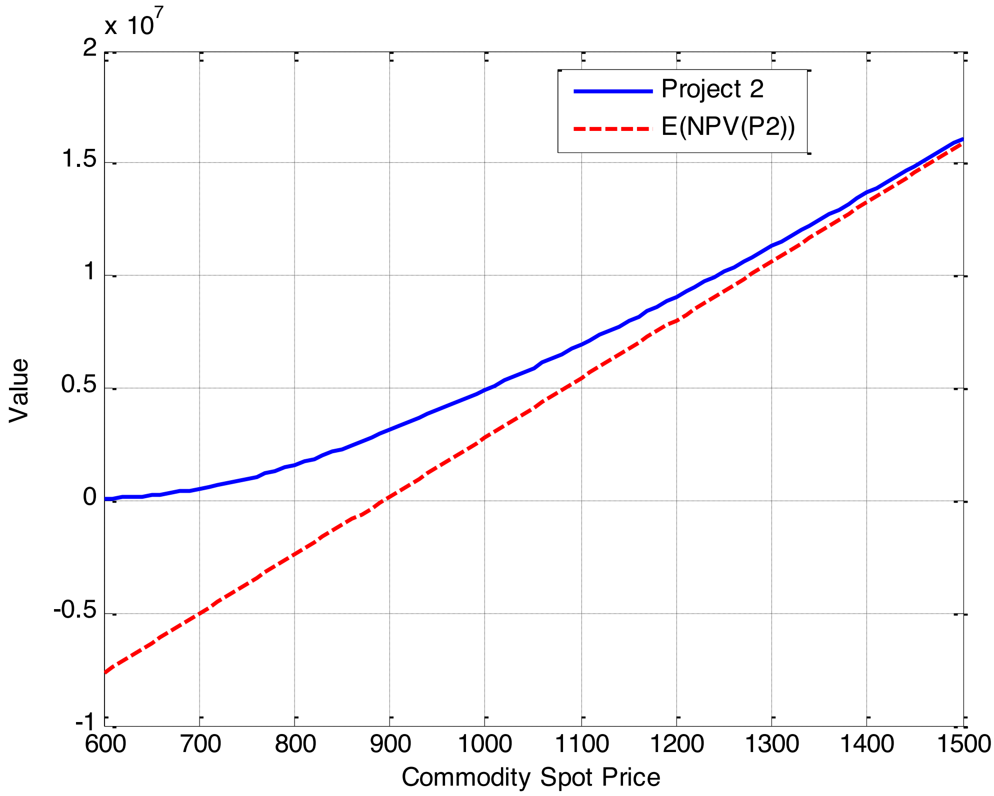

Figure 2. Project 2. Compound mining project with delay, abandonment plus expansion with abandonment / expected NPV.

Figures 1 and 2 show the value of Projects 1 and 2 respectively, calculated by two different methods for different values of the spot commodity price. The blue curves represent the project values using real-option analysis as given by Eq (12) and Eq (18) respectively. The dashed red line represents the value for the projects calculated using the Expected Net Present Value for the different spot values. Clearly the expected NPV calculation ignores the flexibilities available to the mining manager so it must consistently understate the project's value, as can be seen for the range of commodity spot prices in the graphs. The two 'real options' implicit in Eq (12) and the four implicit in Eq (18) are the options to delay and the option to abandon for the basic project and the expansion respectively. As expected, as the spot value increases, the value of the implicit 'real options' go to zero – the value of waiting for more information and the value to forego future cash flows (including costs) will have little value when the spot value is high enough. The value of the options have an inverse relation to the spot price With increasing spot commodity prices therefore we observe the project values asymptotically approaching the values predicted by the Net Present Value considerations.

Figure 3 graphs the values of Project 1 and Project 2 on the same scale. Note how the extra flexibility of Project 2 is reflected in the consistently higher valuation, with the difference lessening as the spot price decreases. Due to the non-linear nature of the 'real option' components of the project values, the marginal value added by the extra flexibility of Project 2 compared to Project 1 is not simply additive but rather diminishes with each extra flexibility. This is consistent with the numerical observations in Trigeorgis (1996).

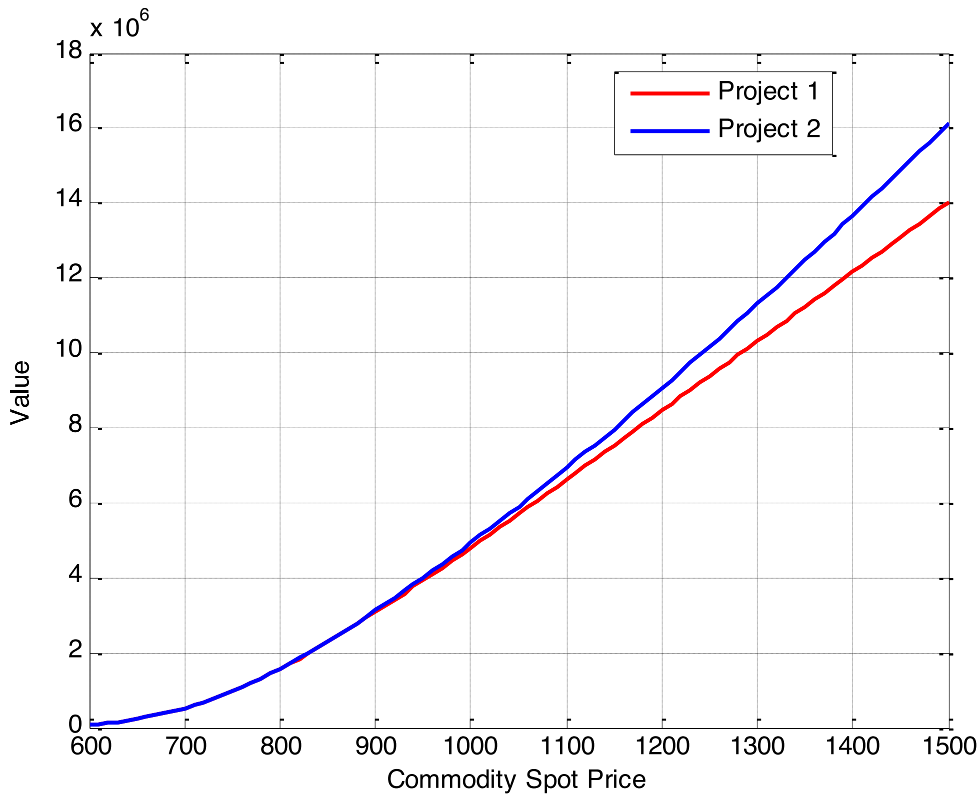

Figure 3. Project 1 and Project 2 on same scale against spot price

CONCLUSION

We have considered the economic valuation of compounded commodity–based mining projects within a tri-expiry framework which we have generalised from the dual-expiry framework of Buchen (2004) to three expiry times. The projects we considered contain several managerial flexibilities, namely the flexibility to delay project commencement, with the further options to abandon production and expand production at later dates. Using the rationale that rising commodity prices would make it economically feasible to expand operations in an already existing project to mine less accessible and more costly ore, we have allowed the added managerial flexibility for optional expanded operations with different (possibly greater) associated fixed costs with their own option to be abandoned independently of the underlying project if conditions deteriorate. We have presented closed-form analytic expressions for the projects solely in terms of first, second and third order Gap instruments. We obtain succinct and novel representations for the value of the projects in terms of option components which are readily interpretable within our methodology as generalised call options, generalised call-on–call options and generalised call-on-call-on-call options respectively on the commodity price. With increasing spot commodity prices we expect the value of the real option component to diminish, and our formulae predict values which asymptotically approach the values predicted by Net Present Value considerations. Extensions of the analysis requiring fourth order and higher instruments are left for a subsequent treatment.

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